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Symmetries of the time-dependent *N*-dimensional oscillator

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Abstract. The study of the symmetry group of the time-dependent oscillator in N dimensions with equation of motion $\ddot{x}_i + \Omega^2(t)x_i = 0$, $i = 1, \ldots, N$, gives the full symmetry group $SL(N+2, \mathbb{R})$ of N^2+4N+3 operators. The Noether subgroup consisting of $\frac{1}{2}(N^2+3N+6)$ operators and the resulting constants of motion are given. A table of the commutation relations between the operators gives the structure constants of the associated Lie algebras.

1. Introduction

Two approaches to the treatment of the symmetries of a dynamical system have been widely used. Noether's theorem (Noether 1918) provides a powerful and well established method of constructing a group of transformations which leaves the action integral invariant (the system under investigation must have a Lagrangian formulation). For each transformation in the group the method enables the determination of a corresponding constant of motion. The other method, utilising Lie's theory of differential equations (Lie 1891, 1922) deals with those transformations which leave the equation of motion invariant. The Lie method in general gives rise to a larger group than the Noether method. However, unlike the latter the former does not appear to yield constants of motion in a straightforward manner.

For the one-dimensional harmonic oscillator given by the equation of motion

$$\ddot{x} + \omega^2 x = 0, \tag{1}$$

where ω is a constant, the Noether method leads to a group of five one-parameter transformations and five associated constants of motion. Of these five constants of motion, only two are functionally independent. The corresponding two transformations form an Abelian subgroup of the Noether group.

Lie's theory of differential equations leads to a 'full' symmetry group of eight one-parameter transformations (Anderson and Davison 1974). It has been identified as $SL(3, \mathbb{R})$ (Wulfman and Wybourne 1976). These authors explored properties of the motion without consideration of the solutions, through the $SO(3)$ subgroup of $SL(3, \mathbb{R})$.

Lutzky (1978a) subsequently investigated the connection between the Noether transformations and the full group of transformations, and found the former to be a five-parameter subgroup of the latter. Moreover, the remaining three transformations of the full group are necessary for the description of certain features of the motion. Thus both approaches are important in the study of the symmetries.

An obvious generalisation of the problem is to the time-dependent oscillator with equation of motion

$$\ddot{x} + \Omega^2(t)x = 0 \tag{2}$$

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where $\Omega(t)$ is a given function of time. Lewis (1968) has derived a conserved quantity for the one-dimensional case by a method other than that of Noether, and it is of interest to construct it using Noether's theorem. Lutzky (1978b) focuses his attention on a particular transformation satisfying Noether's criterion to obtain the Lewis invariant. Eliezer (1978) obtains it along with four other conserved quantities (only two of which are functionally independent, as for the simple harmonic oscillator) in a treatment of the one-dimensional time-dependent oscillator both for the Noether subgroup and the complete symmetry group. In the same paper Eliezer deals with the Noether problem for the time-dependent, three-dimensional oscillator.

This paper deals with some additional features of the problem along with its extension to N dimensions.

We sketch the salient aspects of the time-dependent oscillator previously known in one dimension for clarity. A brief outline of Lie's theory and the calculation of the group generators follows. The next section deals with the Noether subgroup: we obtain the generators by selection from the full group. The constants of motion are also calculated. In the penultimate section we explore the group structure via the associated Lie algebra and identify the full group as $\mathrm{SL}(N+2, \mathbb{R})$. Finally we comment upon the role of the constants in the group structure.

2. The time-dependent oscillator

The oscillator with equation of motion

$$\ddot{x}+\Omega^2(t)x=0 \tag{2}$$

where $\Omega(t)$ is a given function of time has arisen in a variety of problems, an old one being oscillations of a lengthening pendulum (Poe 1845). There has been a revival of interest in this equation since the work of Lewis (1968), who showed that it is possible to construct an invariant for this system, namely

$$I=\tfrac{1}{2}[x^2/\rho^2+(\rho\dot{x}-\dot{\rho}x)^2] \tag{3}$$

where $\rho(t)$ is a function satisfying the 'auxiliary' equation

$$\ddot{\rho}+\Omega^2\rho=\rho^{-3}. \tag{4}$$

Eliezer and Gray (1976) have given a physical interpretation for ρ. The one-dimensional motion given by (2) is considered as the projection of a two-dimensional motion in which the radius vector has length ρ and rotates with angular velocity

$$\dot{\theta}=\rho^{-2}. \tag{5}$$

The relations between the solutions of (2) and (4) have been variously reported (Pinney 1950, Lewis 1968, Eliezer and Gray 1976, Lutzky 1978b, Eliezer 1978). In our case we need to note that if x_1 and x_2 are independent solutions of (2), then the general solution of (4) is given by

$$\rho=(Ax_1^2+Bx_2^2+2Cx_1x_2)^{1/2} \tag{6}$$

where A, B, C are constants such that

$$AB-C^2=W^{-2} \tag{7}$$

where

$$W = x_1\dot{x}_2 - \dot{x}_1 x_2 \tag{8}$$

is the Wronskian (a constant).

Moreover, if $\tilde{\rho}$ is a particular solution of (4), then the general solution of (2) may be written as

$$x = \tilde{\rho}(D\cos\theta + E\sin\theta). \tag{9}$$

3. Lie symmetry groups

The study of the groups of transformations that leave a differential equation invariant was among the pioneering contributions of Sophus Lie. In particular, he considered the invariance of Newton's equation of motion of a free particle.

The main ideas of the theory may be found in Bluman and Cole (1974). An infinitesimal point transformation

$$\bar{t} = t + \delta\alpha\xi(\boldsymbol{x}, t) \qquad \bar{x}_i = x_i + \delta\alpha\eta_i(\boldsymbol{x}, t) \tag{10}$$

is generated by the operator

$$U \equiv \xi(\boldsymbol{x}, t)\frac{\partial}{\partial t} + \eta_i(\boldsymbol{x}, t)\frac{\partial}{\partial x_i}. \tag{11}$$

To see the induced variations in higher derivatives, the n-times extended group operator to be used is

$$U^{(n)} \equiv \frac{\partial}{\partial t} + \eta_i\frac{\partial}{\partial x_i} + \eta_i'\frac{\partial}{\partial \dot{x}_i} + \ldots + \eta_i^{(n)}\frac{\partial}{\partial x_i^{(n)}} \tag{12}$$

where

$$\eta_i^{(k)}(\boldsymbol{x}, \dot{\boldsymbol{x}}, \ldots, \boldsymbol{x}^{(k)}, t) \equiv \frac{\mathrm{d}\eta_i^{(k-1)}}{\mathrm{d}t} - x_i^{(k)}\frac{\mathrm{d}\xi}{\mathrm{d}t} \qquad (k = 1, 2, \ldots, n) \tag{13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \equiv \frac{\partial}{\partial t} + \dot{x}_l\frac{\partial}{\partial x_l} + \ldots + x_l^{(k)}\frac{\partial}{\partial x_l^{(k-1)}}.$$

The finite transformations of the group can be expressed in the form

$$\bar{t} = \mathrm{e}^{\alpha U} t \qquad \bar{\boldsymbol{x}} = \mathrm{e}^{\alpha U}\boldsymbol{x} \qquad \bar{\boldsymbol{x}}^{(k)} = \mathrm{e}^{\alpha U^{(k)}}\boldsymbol{x}^{(k)} \tag{14}$$

(no sum on k) where α is the group parameter. To consider the symmetry group of a differential equation of second order such as

$$\ddot{x}_i + g_i(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) = 0, \tag{15}$$

it is necessary to use the twice-extended operator U'' in the condition

$$U''(\ddot{x}_i + g_i) = 0. \tag{16}$$

In the case of interest

$$\ddot{x}_i + \Omega^2(t)x_i = 0 \qquad (i = 1, \ldots, N) \tag{17}$$

this requirement gives rise to the condition

$$2\xi x_m \Omega\dot{\Omega} + \eta_m \Omega^2 + \eta_{m,tt} - \Omega^2 x_j \eta_{m,j} + 2\Omega^2 x_m \xi_{,t} + \dot{x}_m(-\xi_{,tt} + \Omega^2 x_j \xi_{,j})$$
$$-2\dot{x}_m \dot{x}_j \xi_{,tj} + \dot{x}_j \dot{x}_k \eta_{m,jk} - \dot{x}_m \dot{x}_j \dot{x}_k \xi_{,jk} = 0 \qquad (m = 1, \ldots, N) \tag{18}$$

where

$$\psi_{,j} \equiv \partial\psi/\partial x_j \qquad \psi_{,t} \equiv \partial\psi/\partial t.$$

Equating coefficients of powers of $\dot{x}$ to zero leads to a system of partial differential equations in ξ, η which can be solved to give

$$\xi(x_j, t) = x_k E_k(t) + F(t) \tag{19}$$

$$\eta_i(x_j, t) = x_i x_k \dot{E}_k + \tfrac{1}{2} x_i \dot{F} + A_{ik} x_k + H_i(t) \tag{20}$$

where

$$\ddot{E}_k + \Omega^2 E_k = 0 \qquad \ddot{H}_i + \overset{2}{\Omega} H_i = 0 \tag{21}$$

$$\dddot{F} + 4\Omega\dot{\Omega}F + 4\Omega^2\dot{F} = 0; \tag{22}$$

putting $F = \rho^2$ and integrating we get

$$\ddot{\rho} + \Omega^2 \rho = \rho^{-3}. \tag{4}$$

The A_{km} are arbitrary constants.

Using the theory referred to in § 2, we proceed to calculate the linearly independent group generators

$$G_1 = \rho^2 \sin 2\theta \frac{\partial}{\partial t} + x_j(\rho\dot{\rho} \sin 2\theta + \cos 2\theta)\frac{\partial}{\partial x_j} \tag{23}$$

$$G_2 = \rho^2 \cos 2\theta \frac{\partial}{\partial t} + x_j(\rho\dot{\rho} \cos 2\theta - \sin 2\theta)\frac{\partial}{\partial x_j} \tag{24}$$

$$G_3^k = \rho \cos \theta\, \partial/\partial x_k \tag{25}$$

$$G_4^k = \rho \sin \theta\, \partial/\partial x_k \tag{26}$$

$$G_5 = \rho^2 \frac{\partial}{\partial t} + \rho\dot{\rho} x_j \frac{\partial}{\partial x_j} \tag{27}$$

$$G_6^{jk} = x_j\, \partial/\partial x_k \tag{28}$$

$$G_7^j = x_j \rho \sin \theta \frac{\partial}{\partial t} + x_j(\dot{\rho} \sin \theta + \rho^{-1} \cos \theta) x_k \frac{\partial}{\partial x_k} \tag{29}$$

$$G_8^j = x_j \rho \cos \theta \frac{\partial}{\partial t} + x_j(\dot{\rho} \cos \theta - \rho^{-1} \sin \theta) x_k \frac{\partial}{\partial x_k}, \tag{30}$$

where, as in § 2,

$$\theta(t) = \int_{t_0}^{t} \frac{dt}{\rho^2}. \tag{31}$$

4. Noether's theorem

Noether's theorem gives

$$U'L+\dot{\xi}L-\dot{f}=0 \tag{32}$$

as the condition that the action integral be invariant under a transformation with first extended operator U'. (The function $f=f(x_i, t)$ appears because the Euler–Lagrange equation is unaffected by the addition of a total time derivative to the Lagrangian (cf Lutzky 1978a).) For each transformation satisfying (32) there is an associated constant of motion

$$J=(\xi\dot{x}_k-\eta_k)\frac{\partial L}{\partial \dot{x}_k}-\xi L+f. \tag{33}$$

In our case, where the Lagrangian is taken as

$$L=\tfrac{1}{2}(\dot{x}_k\dot{x}_k-\Omega^2 x_k x_k), \tag{34}$$

then the above condition (32) yields

$$-\xi\Omega\dot{\Omega}x_kx_k-\Omega^2\eta_kx_k-\tfrac{1}{2}\xi_{,t}x_kx_k-f_{,t}+\dot{x}_i(\eta_{i,t}-f_{,i}-\tfrac{1}{2}\Omega^2\xi_{,i}x_kx_k)$$
$$-\tfrac{1}{2}\dot{x}_k\dot{x}_k\xi_{,t}+\dot{x}_i\dot{x}_k\eta_{i,k}-\tfrac{1}{2}\dot{x}_i\dot{x}_i\dot{x}_j\xi_{,j}=0. \tag{35}$$

Proceeding as in § 3, we find that

$$\xi(x_j, t)=F(t) \tag{36}$$

$$\eta_i(x_j, t)=\tfrac{1}{2}x_i\dot{F}+B_{ik}x_k+H_i(t) \tag{37}$$

$$f(x_j, t)=\tfrac{1}{4}x_kx_k\ddot{F}+x_k\dot{H}_k \tag{38}$$

where, for $F=\rho^2$,

$$\ddot{\rho}+\Omega^2\rho=\rho^{-3} \tag{4}$$

and

$$\ddot{H}_k+\Omega^2H_k=0. \tag{39}$$

The B_{km} are arbitrary constants with the restriction

$$B_{km}=-B_{mk}. \tag{40}$$

Comparison of the form of the generators for the Noether group with the form of those for the full group shows that the linearly independent generators here are G_1, G_2, G_3^k, G_4^k, G_5 and $G_6^{[km]}$ where

$$G_6^{[km]}=(G_6^{km}-G_6^{mk}) \tag{41}$$

is twice the antisymmetrised G_6^{km} operator. The commutators are listed in table 1. Each of the generators leads to a constant of motion given by (33). They are

$$J_1=\tfrac{1}{2}[(x_k\dot{\rho}-\dot{x}_k\rho)(x_k\dot{\rho}-\dot{x}_k\rho)\sin 2\theta-x_kx_k\rho^{-2}\sin 2\theta+2x_k\rho^{-1}(x_k\dot{\rho}-\dot{x}_k\rho)\cos 2\theta] \tag{42}$$

$$J_2=\tfrac{1}{2}[(x_k\dot{\rho}-\dot{x}_k\rho)(x_k\dot{\rho}-\dot{x}_k\rho)\cos 2\theta-x_kx_k\rho^{-2}\cos 2\theta-2x_k\rho^{-1}(x_k\dot{\rho}-\dot{x}_k\rho)\sin 2\theta] \tag{43}$$

$$J_3^k = -x_k\rho^{-1}\sin\theta + (x_k\dot\rho - \dot x_k\rho)\cos\theta \tag{44}$$

$$J_4^k = x_k\rho^{-1}\cos\theta + (x_k\dot\rho - \dot x_k\rho)\sin\theta \tag{45}$$

$$J_5 = \tfrac{1}{2}[(x_k\dot\rho - \dot x_k\rho)(x_k\dot\rho - \dot x_k\rho) + x_k x_k\rho^{-2}] \tag{46}$$

$$J_6^{km} = x_m\dot x_k - x_k\dot x_m. \tag{47}$$

Table 1. The commutators $[X_i, X_j]$ for the infinitesimal operators. $G_6^{[ij]} = (G_6^{ij} - G_6^{ij})$, $G_6^{(ij)} = (G_6^{ij} + G_6^{ij})$.

	G_1	G_2	G_3^k	G_4^k	G_5	$G_6^{[kl]}$
G_1	0	$-2G_5$	$-G_3^k$	G_4^k	$-2G_2$	0
G_2	$2G_5$	0	G_4^k	G_3^k	$2G_1$	0
G_3^i	G_3^i	$-G_4^i$	0	0	G_4^i	$(G_3^l\delta_{ik} - G_3^k\delta_{il})$
G_4^i	$-G_4^i$	$-G_3^i$	0	0	$-G_3^i$	$(G_4^l\delta_{ik} - G_4^k\delta_{il})$
G_5	$2G_2$	$-2G_1$	$-G_4^k$	G_3^k	0	0
$G_6^{[ij]}$	0	0	$-(G_3^j\delta_{ik} - G_3^i\delta_{jk})$	$-(G_4^j\delta_{ik} - G_4^i\delta_{jk})$	0	$(-\delta_{ik}G_6^{[jl]} + \delta_{il}G_6^{[jk]}$ $+\delta_{jk}G_6^{[il]} - \delta_{jl}G_6^{[ik]})$
$G_6^{(ij)}$	0	0	$-(G_3^j\delta_{ik} + G_3^i\delta_{jk})$	$-(G_4^j\delta_{ik} + G_4^i\delta_{jk})$	0	$(\delta_{ik}G_6^{(jl)} - \delta_{il}G_6^{(jk)}$ $+\delta_{jk}G_6^{(il)} - \delta_{jl}G_6^{(ik)})$
G_7^i	$-G_7^i$	$-G_8^i$	$-\frac{1}{2}(G_1 + G_6^{mm})\delta_{ik} - G_6^{ik}$	$-\frac{1}{2}(G_5 - G_2)\delta_{ik}$	$-G_8^i$	$-(G_7^k\delta_{il} - G_7^l\delta_{ik})$
G_8^i	G_8^i	$-G_7^i$	$-\frac{1}{2}(G_5 + G_2)\delta_{ik}$	$-\frac{1}{2}(G_1 - G_6^{mm})\delta_{ik} + G_6^{ik}$	G_7^i	$-(G_8^k\delta_{il} - G_8^l\delta_{ik})$

	$G_6^{(kl)}$	G_7^k	G_8^k
G_1	0	G_7^k	$-G_8^k$
G_2	0	G_8^k	G_7^k
G_3^i	$(G_3^l\delta_{ik} + G_3^k\delta_{il})$	$\frac{1}{2}(G_1 + G_6{}^{mm})\delta_{ik} + G_6^{ki}$	$\frac{1}{2}(G_2 + G_5)\delta_{ik}$
G_4^i	$(G_4^l\delta_{ik} + G_4^k\delta_{il})$	$\frac{1}{2}(G_5 - G_2)\delta_{ik}$	$\frac{1}{2}(G_1 - G_6^{mm})\delta_{ik} - G_6^{ki}$
G_5	0	G_8^k	$-G_7^k$
$G_6^{[ij]}$	$(-\delta_{ik}G_6^{(jl)} - \delta_{il}G_6^{(jk)}$ $+\delta_{jk}G_6^{(il)} + \delta_{jl}G_6^{(ik)})$	$(G_7^i\delta_{jk} - G_7^j\delta_{ik})$	$(G_8^i\delta_{jk} - G_8^j\delta_{ik})$
$G_6^{(ij)}$	$(\delta_{ik}G_6^{(jl)} + \delta_{il}G_6^{(jk)}$ $+\delta_{jk}G_6^{(il)} + \delta_{jl}C_6^{(ik)})$	$(G_7^i\delta_{jk} + G_7^j\delta_{ik})$	$(G_8^i\delta_{jk} + G_8^j\delta_{ik})$
G_7^i	$-(G_7^k\delta_{il} + G_7^l\delta_{ik})$	0	0
G_8^i	$-(G_8^k\delta_{il} + G_8^l\delta_{ik})$	0	0

We note that J_3^k and J_4^k are linear constants and that

$$J_1 = J_3^k J_4^k \tag{48}$$

$$J_2 = \tfrac{1}{2}(J_3^k J_3^k - J_4^k J_4^k) \tag{49}$$

$$J_5 = \tfrac{1}{2}(J_3^k J_3^k + J_4^k J_4^k) \tag{50}$$

$$J_6^{km} = J_3^m J_4^k - J_3^k J_4^m. \tag{51}$$

$J_3^i J_4^j + J_3^j J_4^i$ is the Fradkin–Günther–Leach matrix (Günther and Leach 1977).

The complete solution of the problem is essentially provided by G_3^k, G_4^k, which form an Abelian subgroup, and we may write explicitly

$$x_k = \rho(J_4^k\cos\theta - J_3^k\sin\theta). \tag{52}$$

These relations are of the same form for N dimensions as were obtained by Eliezer (1978) for three dimensions.

5. The group structure

Table 1 gives the commutators for the group generators of § 3. Using the relation

$$[X_i, X_j] = C_{ij}^k X_k \tag{53}$$

the structure constants C_{ij}^k of the associated Lie algebra may be read off. The metric tensor of the algebra may be calculated from the formula

$$g_{ij} = C_{ik}^m C_{jm}^k. \tag{54}$$

(In the case of two index operators the appropriate formulae are, for example,

$$[X_{ij}, X_{rs}] = C_{ij,rs}^{mn} X_{mn}. \tag{55}$$

The corresponding metric tensor is given by

$$g_{ij,rs} = C_{ij,pq}^{mn} C_{rs,mn}^{pq}.) \tag{56}$$

Cartan's requirement for semi-simplicity is that the determinant of g_{ij} is non-vanishing (see, for example, Wybourne 1974 or Gilmore 1974). The metric tensor may be shown to satisfy this requirement. Moreover, the metric tensor is indefinite and so the Lie group is non-compact and is a non-compact form of Cartan's A_{N+1} algebra.

The full group has N^2+4N+3 operators whilst the Noether subgroup has only $\frac{1}{2}(N^2+3N+6)$ operators, the difference being $\frac{1}{2}N(N+5)$. Some subgroups of note are the following.

(i) The N SO(3) subgroups (compact): $\{X_1^k, X_2^k, X_3\}$ (k takes a particular value of $1, \ldots, N$ for each subgroup) where

$$X_1^k = G_4^k - G_8^k \tag{57}$$

$$X_2^k = G_3^k + G_7^k \tag{58}$$

$$X_3 = G_5. \tag{59}$$

This corresponds to the SO(3) subgroup which Wulfman and Wybourne (1976) used to investigate the periodicity of the motion. Each has negative definite metric

$$g_{ij} = -2\delta_{ij}.$$

(ii) The subgroup SO(N) corresponding to the $\frac{1}{2}N(N-1)$ 'angular momentum' operators $G_6^{[km]}$. The metric is negative definite, $g_{ij,rs} = -2(N-2)\delta_{ir}\delta_{js}$, $i<j$, $r<s$.

(iii) The Abelian subgroup of G_3^k, G_4^k (fixed k), which, as noted in § 4, essentially describes the motion.

(iv) The two ISO(N) subgroups corresponding to the $\frac{1}{2}N(N-1)$ $G_6^{[km]}$ operators taken with G_3^l, G_4^l respectively. We note that ISO(N) is usually taken as the inhomogeneous Euclidean group of N translations and $\frac{1}{2}N(N-1)$ rotations or another subgroup of the Galilean group: that of N constant-velocity 'boosts' and $\frac{1}{2}N(N-1)$ rotations. However, the N translations/boosts (operators $\partial/\partial x_k$ and $t\partial/\partial x_k$ ($k = 1, \ldots, N$) respectively) are here replaced by $G_3^k = \rho \cos\theta(\partial/\partial x_k)$ or $G_4^k = \rho \sin\theta(\partial/\partial x_k)$ whose finite transformations are

$$\bar{x}_k = x_k + \alpha\rho\cos\theta \qquad \text{and} \qquad \bar{x}_k = x_k + \alpha\rho\sin\theta \tag{60}$$

respectively.

Whilst for the free particle the effect of the boost transformation is to change momentum, the corresponding effect for the harmonic oscillator of G_3^k, G_4^k is a change of amplitude with no change in frequency.

The full group can now be identified as $\mathrm{SL}(N+2, \mathbb{R})$ as it is the only form of the A_{N+1} algebra being both non-compact and with the above subgroups.

In a sense some of these results are to be expected from the work of Wulfman and Wybourne (1976) on the one-dimensional simple harmonic oscillator.

6. Comments

The group structure defined by these equations is the same as that for the constant oscillator for which ρ is a constant. For the constant oscillator in three dimensions the operator G_5 producing the Hamiltonian commutes with the angular momentum operators, and the Hamiltonian is rotationally invariant. The corresponding result for the time-dependent case is that the Lewis constant is rotationally invariant. To pursue this point further we note that

$$G_6'^{[km]}J_5 = 0 \tag{61}$$

where $G_6'^{[km]}$ is the first extended operator of $G_6^{[km]}$. In general, if J_i is the constant of motion produced from (33) by a transformation X_i satisfying (32), then

$$X_i'J_i = 0 \qquad \text{(no sum on } i\text{)}. \tag{62}$$

The interpretation of (62) is that as the operator transforms one solution into another the value of J_i is unchanged.

For the Noether subgroup (for which constants of motion are available) it is of interest to note that some relations between the operators and the constants reflect the structure of the subgroup, for example

$$G_3'^k J_4^l = \delta_{kl} \qquad G_4'^k J_3^l = -\delta_{kl}. \tag{63}$$

Considering the one-dimensional case, we write in $(t, x, \dot{x})$ space

$$G_3' J_4 = (\boldsymbol{T}^{(3)}.\boldsymbol{\nabla})J_4 = 1 \qquad \text{and} \qquad (\boldsymbol{T}^{(3)}.\boldsymbol{\nabla})J_3 = 0 \tag{64}$$

where

$$\boldsymbol{\nabla} \equiv (\partial/\partial t, \partial/\partial x, \partial/\partial \dot{x}) \qquad \text{and} \qquad \boldsymbol{T}^{(3)} = (0, \rho\cos\theta, \dot{\rho}\cos\theta - \rho^{-1}\sin\theta).$$

Similarly,

$$\boldsymbol{T}^{(4)}.\boldsymbol{\nabla}J_3 = 1 \qquad \boldsymbol{T}^{(4)}.\boldsymbol{\nabla}J_4 = 0. \tag{65}$$

The relations (64) and (65) indicate the functional independence of the constants J_3 and J_4.

We may use (63) to calculate $G_3'^k J_1, \ldots, G_3'^k J_6^{lm}$ and similarly for $G_4'^k$. We get, using (47)–(51),

$$\begin{aligned} G_3'^k J_1 &= J_3^k & G_4'^k J_1 &= -J_4^k \\ G_3'^k J_2 &= -J_4^k & G_4'^k J_2 &= -J_3^k \\ G_3'^k J_5 &= J_4^k & G_4'^k J_5 &= -J_3^k \\ G_3'^k J_6^{lm} &= J_3^m \delta_{kl} - J_3^l \delta_{km} & G_4'^k J_6^{lm} &= J_4^m \delta_{kl} - J_4^l \delta_{km}. \end{aligned} \tag{66}$$

Then using these, (62), (47)–(51) and noting that the operators X_i and their extensions X_i' obey the same commutation relations we can calculate all other such relations for the Noether subgroup. We find

$$X_i'J_k = C_{ik}^l J_l \qquad (i, k \neq 3, 4) \tag{67}$$

where the X_i are members of the Noether subgroup and the C_{ik}^l are the structure constants.

The constants provided by the Noether subgroup play an important auxiliary role in the study of the group structure. It would seem desirable to have constants of motion corresponding to the other transformations of the full group. A method of calculating these constants and investigation of their significance will be the subject of a further paper.

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